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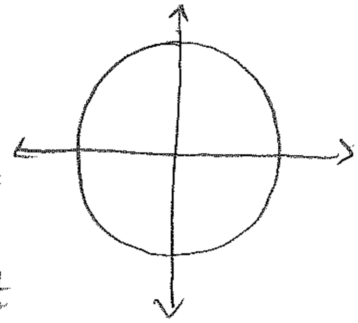
Implicit Differentiation

$x^2+y^2=1$

What is the slope of the tangent line to the unit circle at any point?



We are really good at finding derivatives of functions, So we could break the circle into two functions:

$$f_1(x)=\sqrt{1-x^2}$$
$$f_2(x)=-\sqrt{1-x^2}$$

But this is messy and, many times we won't be able to split such a relationship into different functions.

we will obtain derivative of this relationship between x and y by using Implicit Differentiation (which we will see is just an application of the Chain Rule.

An Implicitly Defined Function:

Explicit: $f(x)=$ stuff w/ x

Implicit: $F(x, f(x))=0$ ← a lot of times we have a constant here.

- F is some expression w/ x and f(x)
- most of the time we write y in place of f(x). think of them as the same thing.

Ex: (implicit Functions)

$x^3-8xy+y^2=1$

To have explicit func, would have to solve for y. But that's hard!

$\sin(xy^2)-2y^3+5x=2$

Steps of Implicit Differentiation

Step 1: take derivative of both sides

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)$$

$$\Updownarrow$$

$$\underbrace{\frac{d}{dx}(x^2)}_{2x}+\underbrace{\frac{d}{dx}y^2}_{\frac{dy^2}{dy}\cdot\frac{dy}{dx}}=0$$

$$\frac{dy^2}{dy}\cdot\frac{dy}{dx} = 2y\cdot\frac{dy}{dx}$$

apply the Chain Rule!

Since $y = f(x)$

taking derivative of an expression w/ y, requires us to use the chain rule.

$$\frac{d}{dx}(y^2) = 2y\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}(f(x))^2 = 2f(x)\cdot f'(x)$$

$$2x+2y\frac{dy}{dx}=0$$

Step 2: Solve for $\frac{dy}{dx}$.

$$2y\frac{dy}{dx}=-2x$$

$$\frac{dy}{dx}=\frac{-2x}{2y}=-\frac{x}{y}$$

Step 3: use your result.

What is the slope of tangent line at $(1/2, \sqrt{3}/2)$?

$$\left.\frac{dy}{dx}\right|_{(1/2,\sqrt{3}/2)} = -\frac{1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

Important Note: $\frac{dy}{dx}$ is usually what you're looking for. We could find $\frac{dx}{dy}$

(think of x as a function of y)

• variables don't always need to be x and y (HW has s, t)

Ex: $x^2+y^2=1 \Leftrightarrow 2x\frac{dx}{dy}+2y=0 \Leftrightarrow \frac{dx}{dy}=-\frac{y}{x}$. In general, $\boxed{\frac{dy}{dx}=\frac{1}{(dx/dy)}}$

Ex: What is the slope of tangent line to $x^3-8xy+y^2=1$ at $(1,0)$

(1) $\frac{d}{dx}(x^3-8xy+y^2)=\frac{d}{dx}(1)$

$$3x^2-\underbrace{\left(8y+8x\frac{dy}{dx}\right)}_{\text{product rule!}}+2y\frac{dy}{dx}=0$$

(2) $3x^2-8y-8x\frac{dy}{dx}+2y\frac{dy}{dx}=0$

$$3x^2-8y=8x\frac{dy}{dx}-2y\frac{dy}{dx}$$

$$3x^2-8y=\frac{dy}{dx}(8x-2y)$$

$$\frac{3x^2-8y}{8x-2y}=\frac{dy}{dx}$$

(3) $\left.\frac{dy}{dx}\right|_{(1,0)}=\frac{3(1)^2-8(0)}{8(1)-2(0)}=\frac{3}{8}$

Multiple Derivatives

Ex: $xy^2 = 3$. Find 1st and 2nd derivative.

(1) $\frac{d}{dx}(\underbrace{xy^2}) = \frac{d}{dx}(3)$

$$y^2 + 2xy\frac{dy}{dx} = 0$$

(2) $-\frac{y^2}{2xy} = \frac{dy}{dx}$ $=$ $\frac{-y}{2x}$

First der: $\frac{dy}{dx} = -\frac{y}{2x}$

Second Derivative:

$$\frac{dy}{dx} = \frac{-y}{2x}$$

(1) $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-y}{2x}\right)$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-y)(2x)^{-1}$$

$$= (-1)\left(\frac{dy}{dx}\right)(2x)^{-1} + (-y)(-1)(2x)^{-2} \cdot (2)$$

$$= -\underbrace{\frac{dy}{dx}} \cdot \frac{1}{2x} + \frac{2}{(2x)^2}$$

need to sub in the derivative $\frac{dy}{dx} = \frac{-y}{2x}$

$$= -\left(\frac{-y}{2x}\right) \cdot \frac{1}{2x} + \frac{2}{(2x)^2}$$

$$= \frac{y+2}{(2x)^2}$$

* derivative of inverse